INAR models with dynamic coefficient driven by a SRE

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Outline

Background (1)

- **INteger-valued AutoRegressive (INAR)** models are one of the most popular models for count time series.

- The **thinning operator** "$\circ$" satisfies

$$\alpha \circ N \sim Bin(N, \alpha),$$

for an $N \in \mathbb{N}$ and $\alpha \in (0, 1)$.

- The **first order INAR** model is given by

$$y_t = \alpha \circ y_{t-1} + \varepsilon_t, \; t \in \mathbb{Z},$$

where $\{\varepsilon_t\}_{t \in \mathbb{Z}}$ is an iid sequence of count variables.

Background (2)

- Real time series data often exhibit changing dynamic behaviors and the classic INAR model may not be able to properly handle these situations.

- We introduce a class of INAR models that allows the survival probability α_t to be updated at each time period using past information.

- Time varying survival probabilities have been also considered by Zheng *et al.* (2007) and Zheng and Basawa (2008). The latter authors specify the survival probability as logit $\alpha_t = \omega + \tau y_{t-1}$.

INAR models with dynamic α_t

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- We allow the survival probability α to vary over time relying on the **GAS framework** of Creal *et al.* (2013) and Harvey (2013).

- The **GAS-INAR** model is described by the following equations

$$\begin{aligned} y_t =& \alpha_t \circ y_{t-1} + \varepsilon_t, \\ \operatorname{logit} \alpha_{t+1} =& \omega + \beta \operatorname{logit} \alpha_t + \tau s_t, \end{aligned}$$

- The innovation s_t is the score of the predictive likelihood, i.e.

$$s_t = \partial \log p(y_t | y_{t-1}, \alpha_t) / \partial \operatorname{logit} \alpha_t,$$

where $p(y_t|\alpha_t, y_{t-1})$ is the pmf of y_t given y_{t-1} and α_t.

ML estimation

- ML estimation can be easily performed as the likelihood function is available in closed form.

- Using the data $\{y_t\}_{t=1}^T$, the **filtered probability** is obtained as

$$\text{logit}\, \hat{\alpha}_{t+1}(\theta) = \omega + \beta\, \text{logit}\, \hat{\alpha}_t(\theta) + \tau \hat{s}_t,$$

  where $\text{logit}\, \hat{\alpha}_1(\theta) = \omega/(1-\beta)$.

- Then, the MLE $\hat{\theta}_T$ is the maximizer of the likelihood function

$$\hat{L}_T(\theta) = \sum_{t=2}^{T} \log p(y_t | \hat{\alpha}_t(\theta), y_{t-1}),$$

  over the parameter set Θ.

Estimation under misspecification

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- The **GAS-INAR** model should not be considered a Data Generating Process (DGP) but a filter to approximate an unknown DGP (Blasques *et al.*, 2015).

- The conditional **KL divergence** is given by

$$KL_t(\theta) = \sum_{x=0}^{\infty} \log \left(\frac{p^o(x|I_{t-1})}{p(x|\tilde{\alpha}_t(\theta), y_{t-1})} \right) p^o(x|I_{t-1}),$$

  where $p^o(x|I_{t-1})$ is the true conditional pmf of the observations.

- The **pseudo-true** parameter θ^* is defined as the minimizer of the average KL divergence $\mathbb{E}KL_t(\theta)$ in the parameter set Θ.

Consistency of ML estimation

(C.1) The DGP $\{y_t\}_{t \in \mathbb{Z}}$ is stationary and ergodic count process.

(C.2) The moments condition $Ey_t^2 < \infty$ is satisfied.

(C.3) The compact set Θ is such that $E \log \Lambda_t(\theta) < 0, \ \forall \ \theta \in \Theta$.

(C.4) The model is identifiable in the compact set Θ.

Theorem

Let conditions (C.1)-(C.4) hold, then the MLE $\hat{\theta}_T$ is strongly consistent

$$\hat{\theta}_T \xrightarrow{a.s.} \theta^*, \quad T \to \infty.$$

Simulation: DGPs

- Consider DGPs of the form

$$y_t = \alpha_t^o \circ y_{t-1} + \varepsilon_t, \;\; \varepsilon_t \sim \mathcal{P}(5),$$

where the DGPs differ for the specification of α_t^o.

- The following four dynamics are considered:

(1) **Fast sine**: $\alpha_t^o = 0.5 + 0.25 \sin(\pi t/100)$.
(2) **Slow sine**: $\alpha_t^o = 0.5 + 0.25 \sin(\pi t/250)$.
(3) **Fast steps**: $\alpha_t^o = 0.25 I_{[-1,0]}(\sin(\pi t/100)) + 0.75 I_{(0,1]}(\sin(\pi t/100))$.
(4) **Slow steps**: $\alpha_t^o = 0.25 I_{[-1,0]}(\sin(\pi t/250)) + 0.75 I_{(0,1]}(\sin(\pi t/250))$.

Simulations: filtered paths for α_t

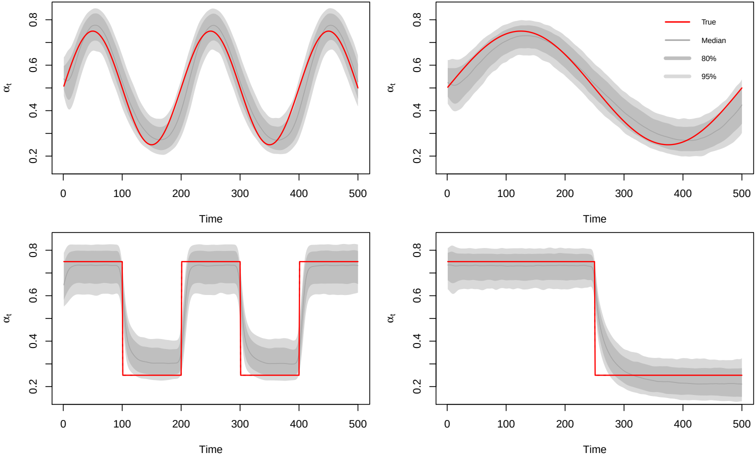

Figure: Confidence bounds for the filtered paths of the surviving probability.

Simulation: tables

	Point prediction (MSE)			
	Fast sine	Slow sine	Fast steps	Slow steps
INAR	0.242	0.257	0.322	0.356
rc-INAR	0.112	0.111	0.145	0.132
GAS-INAR	**0.077**	**0.060**	**0.101**	**0.072**

	Pmf prediction (KL divergence)			
	Fast sine	Slow sine	Fast steps	Slow steps
INAR	0.238	0.253	0.412	0.442
rc-INAR	0.117	0.114	0.212	0.185
GAS-INAR	**0.053**	**0.029**	**0.128**	**0.057**

Table: *MSE and KL divergence between the true DGP and the different models. The rc-INAR model is the model of Zheng and Basawa (2008).*

Crime time series

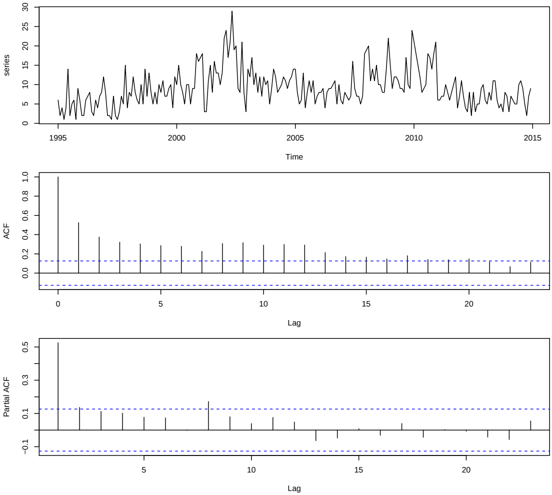

Figure: Monthly number of criminal reports in Blacktown, Australia, with empirical autocorrelation functions.

	ω	β	τ	μ	σ^2	log-lik	pvalue	AIC
GAS-NBINAR	-0.907 (0.338)	0.965 (0.027)	0.135 (0.055)	6.083 (0.481)	14.155 (1.853)	-662.91	0.002	1335.82
NBINAR	-0.401 (0.176)	-	-	5.586 (0.456)	15.265 (2.125)	-669.03	-	1344.07
GAS-PoINAR	-1.258 (0.294)	0.967 (0.019)	0.141 (0.033)	6.539 (0.313)	-	-695.04	0.000	1398.24
PoINAR	-0.613 (0.140)	-	-	6.046 (0.323)	-	-714.58	-	1433.21

Table: *ML estimate for different specifications.*

Filtered survival probability

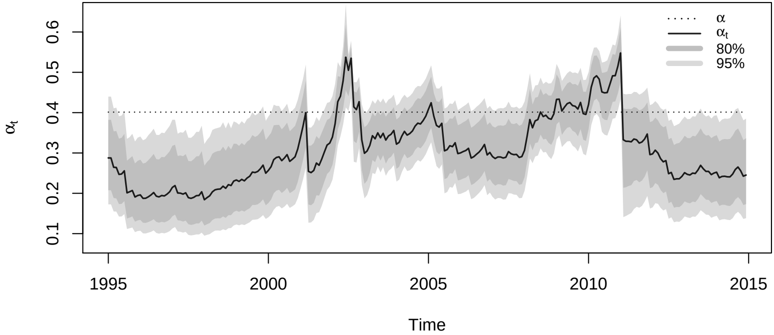

Figure: Filtered survival probability from the GAS-NBINAR model with confidence bounds.

Out-of-sample results

	Point forecasts (MSE)					
	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$h = 6$
GAS-NBINAR	**15.77**	**20.15**	**20.56**	**21.51**	**21.36**	**21.23**
NBINAR	16.51	21.47	22.61	23.70	23.85	23.72
GAS-PoINAR	16.33	20.66	21.18	21.98	21.82	21.52
PoINAR	17.00	21.82	22.86	23.79	23.91	23.78
	Pmf forecasts (log score)					
	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$h = 6$
GAS-NBINAR	**-2.73**	**-2.82**	**-2.83**	**-2.85**	**-2.85**	**-2.85**
NBINAR	-2.75	-2.85	-2.88	-2.91	-2.91	-2.91
GAS-PoINAR	-2.83	-2.96	-2.98	-3.00	-3.00	-2.98
PoINAR	-2.88	-3.08	-3.12	-3.18	-3.19	-3.18

Table: *Forecast MSE and log score criterion computed in the last 100 observations for different forecast horizons h.*

Concluding comments

- We provide a dynamic specification for the INAR survival probability based on the score framework of Creal et al. (2013) and Harvey (2013).
- The model should not be interpreted as a DGP but as a filter. In this direction, we show the consistency of ML estimation.
- Simulation and empirical experiments illustrate the flexibility and usefulness of the proposed model.

Main references

Blasques, F., Koopman, S. J., and Lucas, A. (2015). Information-theoretic optimality of observation-driven time series models for continuous responses. *Biometrika* 102:325–343.

Creal, D., Koopman, S. J., and Lucas, A. (2013). Generalized autoregressive score models with applications.. *Journal of Applied Econometrics*, 28(5):777–795.

Harvey (2013). Dynamic Models for Volatility and Heavy Tails: With Applications to Financial and Economic Time Series. *New York: Cambridge University Press.*